American Journal of Science, Engineering and Technology
2017; 2(1): 15-19
http://www.sciencepublishinggroup.com/j/ajset
doi: 10.11648/j.ajset.20170201.13

Winds Generated by Flows and Riemannian Metrics

Savin Treanţă, Elena-Laura Dudaş

Faculty of Applied Sciences, University "Politehnica" of Bucharest, Bucharest, Romania

Email address:
savin_treanta@yahoo.com (S. Treanţă), laura.otobicu@gmail.com (Elena-Laura Dudaş)

To cite this article:
Savin Treanţă, Elena-Laura Dudaş. Winds Generated by Flows and Riemannian Metrics. *American Journal of Science, Engineering and Technology*. Vol. 2, No. 1, 2017, pp. 15-19. doi: 10.11648/j.ajset.20170201.13

Received: December 22, 2017; **Accepted**: January 9, 2017; **Published**: January 24, 2017

Abstract: The winds theory is based on PDEs whose unknown is the velocity vector field depending on time and spatial coordinates. The geometric dynamics is formulated using ODEs associated to a flow and a Riemannian metric, where the unknown is the velocity vector field depending on time. In this paper, we join these ideas showing that some geometric dynamics models generate winds. The second part of this paper is focused on the stability analysis of the considered models.

Keywords: Flow, Metric, Geometric Dynamics, Wind, Stability

1. Introduction

The *wind* usually refers to the horizontal component of the air motion relative to the earth. Mathematically, the wind is represented by the velocity vector field $\vec{v}(x, y, z, t)$, where $(x, y, z) \in \mathbb{R}^3$ and $t \in \mathbb{R}$. The forces applied to an element of air, moving almost horizontally over the earth's surface with velocity $\vec{v}$, are:

(1) the inertial force, represented by the acceleration $\frac{\mathrm{D}\vec{v}}{\mathrm{Dt}} = \frac{\partial \vec{v}}{\partial \mathrm{t}} + (\vec{v}\nabla)\vec{v}$;

(2) the pressure gradient force, represented by $-\frac{1}{\rho}\operatorname{grad} p$;

(3) the deviating force due to the earth's rotation, represented by $\vec{v} \times \vec{F}_1$;

(4) the shearing stresses produced by the relative motion of the layers above and below, represented by $\vec{F}_2$.

Consequently, the partial differential equation of motion is of the form

$$\frac{\mathrm{D}\vec{v}}{\mathrm{Dt}} = -\frac{1}{\rho}\operatorname{grad} p - \vec{v} \times \vec{F}_1 + \vec{F}_2 \tag{1}$$

and the winds are classified according to the relative importance of the four terms from the above relation:

Geostrophic wind: The motion is stationary (i.e. it is independent by t) and only terms (2) and (3) remain. The wind is expressed in terms of the horizontal gradient pressure and is along the isobars.

Gradient wind: The term (1) is assumed to be approximately equal to v^2/R, where R is the radius of curvature of the isobars. The acceleration in the direction of motion is ignored. The wind is given in terms of the pressure gradient by a quadratic equation. The term (4) is omitted.

Antitriptic wind: The terms (2) and (4) are dominant and (4) represents the friction at the ground and it is therefore a force in the opposite direction to the motion. The wind is towards low pressure.

Ageostrophic wind: Departure from the geostrophic wind may be produced in a variety of ways in frictionless motion. The term (4) is dominant and it expresses the convection at the earth's surface.

The geometric dynamics (see [8, 9, 10]) is similar with the winds theory (see [6, 10, 4]): instead of the velocity vector field $\vec{v}(x, y, z, t)$, where $(x, y, z) \in \mathbb{R}^3$ and $t \in \mathbb{R}$, we work with the velocity vector field $\vec{v}(t)$ along a curve. In this case, the term $(\vec{v}\nabla)\vec{v}$ doesn't exist in the total acceleration, $\frac{\mathrm{D}\vec{v}}{\mathrm{Dt}} = \frac{\partial \vec{v}}{\partial \mathrm{t}} + (\vec{v}\nabla)\vec{v}$. For other connected viewpoints on this subject, the reader is directed to [1, 2, 3, 5, 7].

2. Winds and Geometric Dynamics

2.1. Hopf Geometric Dynamics and Hopf Wind

Consider the Riemannian manifold $(\mathbb{R}^2, \delta_{ij})$ (see δ_{ij} as the canonical (usual) metric in $\mathbb{R}^2$; δ_{ij} = Kronecker's symbol) and the non-linear system of differential equations

$$\frac{dx_1}{dt} = -x_2 + x_1[\lambda - (x_1^2 + x_2^2)], \frac{dx_2}{dt} = x_1 + x_2[\lambda - (x_1^2 + x_2^2)], \tag{2}$$

which describes a *bifurcation of Hopf type*. Let $X = (X_1, X_2)$ be a vector field on $\mathbb{R}^2$, where

$$X_1(x_1,x_2) = -x_2 + x_1[\lambda - (x_1^2 + x_2^2)], \quad (3)$$
$$X_2(x_1,x_2) = x_1 + x_2[\lambda - (x_1^2 + x_2^2)]$$

and let $f(x_1,x_2) = \frac{1}{2} \parallel X \parallel^2$ be the energy of the vector field X. By a direct computation, we get $\operatorname{rot} X = (0,0,2)$ and $\operatorname{div} X = 2\lambda - 3(x_1^2 + x_2^2)$.

The Hopf geometric dynamics is described by

$$\frac{d^2x_i}{dt^2} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{2}\left(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}\right)\frac{dx_j}{dt}, i = 1,2, \quad (4)$$

or

$$\frac{d^2x_1}{dt^2} = -2\frac{dx_2}{dt} + \frac{\partial f}{\partial x_1}, \frac{d^2x_2}{dt^2} = 2\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}. \quad (5)$$

Now, let us explain how a wind is produced by the Hopf flow and by the Euclidean metric. Firstly, we extend the Hopf vector field from $\mathbb{R}^2$ to $\mathbb{R}^3$. In this sense, we introduce

$$X = (X_1, X_2, X_3), X_1(x_1,x_2,x_3) = -x_2 + x_1[\lambda - (x_1^2 + x_2^2)], X_2(x_1,x_2,x_3) = x_1 + x_2[\lambda - (x_1^2 + x_2^2)], X_3(x_1,x_2,x_3) = 0.$$

The vector field $\operatorname{rot} X = (0,0,2)$ can be written as $\operatorname{grad} \varphi$, where $\varphi(x_1,x_2,x_3) = 2x_3$. Consequently, in vector notation, the wind produced by the Hopf flow and by the Euclidean metric is described by the second-order equation

$$\frac{d^2x}{dt^2} = \operatorname{grad} f + \operatorname{grad} \varphi \times \frac{dx}{dt}. \quad (6)$$

2.2. Rabinovich Geometric Dynamics and Rabinovich Wind

Here, we use the Riemannian manifold $(\mathbb{R}^3, \delta_{ij})$ and consider the following non-linear system of differential equations

$$\frac{dx_1}{dt} = x_2x_3, \frac{dx_2}{dt} = -x_1x_3, \frac{dx_3}{dt} = x_1x_2, \quad (7)$$

which is known as *Rabinovich-type system*. Let $X = (X_1, X_2, X_3)$ be a vector field on $\mathbb{R}^3$, where

$$X_1(x_1,x_2,x_3) = x_2x_3, X_2(x_1,x_2,x_3) = -x_1x_3, X_3(x_1,x_2,x_3) = x_1x_2 \quad (8)$$

and let $f(x_1,x_2,x_3) = \frac{1}{2} \parallel X \parallel^2$ be the energy of the vector field X. By direct computation, we obtain $\operatorname{rot} X = (2x_1, 0, -2x_3)$ and $\operatorname{div} X = 0$.

Remark 2.2.1 It is well-known that the divergence of a vector field defines the speed of contraction-dilation of the volumes by the flow generated by the vector field. In our case, we obtained $\operatorname{div} X = 0$, so X is a solenoidal vector field, that is, the Rabinovich flow conserves the areas.

The Rabinovich geometric dynamics is described by

$$\frac{d^2x_i}{dt^2} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{3}\left(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}\right)\frac{dx_j}{dt}, i = 1,2,3, \quad (9)$$

or

$$\frac{d^2x_1}{dt^2} = 2x_3\frac{dx_2}{dt} + \frac{\partial f}{\partial x_1}, \frac{d^2x_2}{dt^2} = -2x_3\frac{dx_1}{dt} - 2x_1\frac{dx_3}{dt} + \frac{\partial f}{\partial x_2}, \quad (10)$$
$$\frac{d^2x_3}{dt^2} = 2x_1\frac{dx_2}{dt} + \frac{\partial f}{\partial x_3}.$$

Let us introduce the wind produced by the Rabinovich flow and by the Euclidean metric. We remark that $\operatorname{rot} X = (2x_1, 0, -2x_3)$ can be written as $\operatorname{grad}\varphi$, where $\varphi(x_1,x_2,x_3) = x_1^2 - x_3^2$. So, in vector notation, the wind produced by the Rabinovich flow and by the Euclidean metric is given by the second-order equation

$$\frac{d^2x}{dt^2} = \operatorname{grad} f + \operatorname{grad} \varphi \times \frac{dx}{dt}. \quad (11)$$

2.3. Van der Pol Geometric Dynamics and Van Der Pol Wind

Let us start with the Riemannian manifold $(\mathbb{R}^2, \delta_{ij})$ and consider the following non-linear system of differential equations

$$\frac{dx_1}{dt} = x_2, \frac{dx_2}{dt} = -\varepsilon(x_1^2 - 1)x_2 - x_1, \quad (12)$$

which is known as *van der Pol oscillator*, ε being a control parameter. Let $X = (X_1, X_2)$ be a vector field on $\mathbb{R}^2$, with

$$X_1(x_1,x_2) = x_2, X_2(x_1,x_2) = -\varepsilon(x_1^2 - 1)x_2 - x_1 \quad (13)$$

and let $f(x_1,x_2) = \frac{1}{2} \parallel X \parallel^2$ be the energy of the vector field X. By a simple calculation, we get $\operatorname{rot} X = (0,0,-2\varepsilon x_1x_2 - 2)$ and $\operatorname{div} X = -\varepsilon x_1^2 + \varepsilon$.

The van der Pol geometric dynamics is described by

$$\frac{d^2x_i}{dt^2} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{2}\left(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}\right)\frac{dx_j}{dt}, i = 1,2, \quad (14)$$

or

$$\frac{d^2x_1}{dt^2} = (2 + 2\varepsilon x_1x_2)\frac{dx_2}{dt} + \frac{\partial f}{\partial x_1}, \frac{d^2x_2}{dt^2} = (-2 - 2\varepsilon x_1x_2)\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}. \quad (15)$$

The wind produced by the van der Pol flow and by the Euclidean metric is determined as follows. We extend the van der Pol vector field from $\mathbb{R}^2$ to $\mathbb{R}^3$. In this direction, we introduce the vector field $X = (X_1, X_2, X_3)$, with $X_1(x_1,x_2,x_3) = x_2, X_2(x_1,x_2,x_3) = -\varepsilon(x_1^2 - 1)x_2 - x_1, X_3(x_1,x_2,x_3) = 0$. The vector field $Y := (0,0,-2\varepsilon x_1x_2 - 2)$ can be written using the Monge representation $Y = \operatorname{grad} f_3 \times f_1 \operatorname{grad} f_2$. Thus, in vector notation, the wind produced by the van der Pol flow and by the Euclidean metric is given by the second-order equation

$$\frac{d^2x}{dt^2} = \operatorname{grad} f + Y \times \frac{dx}{dt}. \quad (16)$$

2.4. Phytoplankton Geometric Dynamics and Phytoplankton Wind

Further, we shall consider a biological model. We also use the Riemann manifold $(\mathbb{R}^3, \delta_{ij})$. Define the non-linear system of differential equations

$$\frac{dx_1}{dt} = 1 - x_1 - \frac{x_1 x_2}{4}, \frac{dx_2}{dt} = (2x_3 - 1)x_2, \frac{dx_3}{dt} = \frac{x_1}{4} - 2x_3^2, \quad (17)$$

which describes the *Phytoplankton Growth Model*, where x_1 is the substrate, x_2 is the phytoplankton biomass and x_3 is the intracellular nutrient per biomass.

Let $X = (X_1, X_2, X_3)$ be a vector field on $\mathbb{R}^3$, where

$$X_1(x_1, x_2, x_3) = 1 - x_1 - \frac{x_1 x_2}{4}, X_2(x_1, x_2, x_3) = (2x_3 - 1)x_2, \quad (18)$$

$$X_3(x_1, x_2, x_3) = \frac{x_1}{4} - 2x_3^2$$

and let $f(x_1, x_2, x_3) = \frac{1}{2} \parallel X \parallel^2$ be the energy of the vector field X. By simple computations, we get $\operatorname{rot} X = \left(-2x_2, -\frac{1}{4}, -\frac{x_1}{4}\right)$ and $\operatorname{div} X = -2 - \frac{x_2}{4} - 2x_3$.

The Phytoplankton geometric dynamics is described by

$$\frac{d^2 x_i}{dt^2} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{3} \left(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}\right) \frac{dx_j}{dt}, i = 1,2,3, \quad (19)$$

or

$$\frac{d^2 x_1}{dt^2} = -\frac{x_1}{4}\frac{dx_2}{dt} - \frac{1}{4}\frac{dx_3}{dt} + \frac{\partial f}{\partial x_1}, \frac{d^2 x_2}{dt^2} = \frac{x_1}{4}\frac{dx_1}{dt} + 2x_2\frac{dx_3}{dt} + \frac{\partial f}{\partial x_2}, \quad (20)$$

$$\frac{d^2 x_3}{dt^2} = \frac{1}{4}\frac{dx_1}{dt} - 2x_2\frac{dx_2}{dt} + \frac{\partial f}{\partial x_3}.$$

Let us introduce the wind produced by the Phytoplankton flow and by the Euclidean metric. The vector field $V := \left(-2x_2, -\frac{1}{4}, -\frac{x_1}{4}\right)$ can be written using the Monge representation $V = \operatorname{grad} f_3 \times f_1 \operatorname{grad} f_2$. So, in vector notation, the wind produced by the Phytoplankton flow and by the Euclidean metric is described by the second-order equation

$$\frac{d^2 x}{dt^2} = \operatorname{grad} f + V \times \frac{dx}{dt}. \quad (21)$$

3. Stability Analysis of Considered Models

3.1. Stability Analysis of Hopf Bifurcation

The equilibrium point $\left(x_1^*(t), x_2^*(t)\right)$ of the bifurcation of Hopf type is the solution of the following algebraic system

$$-x_2 + x_1[\lambda - (x_1^2 + x_2^2)] = 0, x_1 + x_2[\lambda - (x_1^2 + x_2^2)] = 0, \quad (22)$$

where λ is a parameter. In this case, we get $\left(x_1^*(t), x_2^*(t)\right) = (0,0)$. Denoting

$$x_1 = x_1^* + \varepsilon y_1, x_2 = x_2^* + \varepsilon y_2, \quad (23)$$

the linearization around the equilibrium point $(0,0)$ is

$$\dot{y} = \begin{pmatrix} \lambda - 3x_1^2 - x_2^2 & -1 - 2x_1 x_2 \\ 1 - 2x_1 x_2 & \lambda - 3x_2^2 - x_1^2 \end{pmatrix}_{(x_1^*, x_2^*)} \cdot y, \quad (24)$$

that is,

$$\dot{y} = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} \cdot y, \quad (25)$$

where $y = (y_1, y_2)$ is defined as in the previous. By direct computation, we get $\det(A - sI_2) = s^2 - 2\lambda s + \lambda^2 + 1 = s^2 - \operatorname{Tr}(A)s + \det(A)$ (see A as the Jacobian matrix of the function $(-x_2 + x_1[\lambda - (x_1^2 + x_2^2)], x_1 + x_2[\lambda - (x_1^2 + x_2^2)])$, computed at $\left(x_1^*(t), x_2^*(t)\right) = (0,0)$). Solving the equation $\det(A - sI_2) = 0$, we obtain the solutions $s_1 = \lambda + i, s_2 = \lambda - i$. Consequently, for $\lambda > 0$ the previous linearized system is unstable, for $\lambda < 0$ is asymptotically stable, and at $\lambda = 0$ (i.e. $s_{1,2} = \pm i$) we find $m_a(\pm i) = m_g(\pm i) = 1$, so the linearized system is stable (see $m_a(\alpha), m_g(\alpha)$ as algebraic multiplicity, respectively geometric multiplicity of α).

Let us summarize the previous analysis as

Proposition 3.1.1 *For the non-linear system of differential equations (bifurcation of Hopf type)*

$$\frac{dx_1}{dt} = -x_2 + x_1[\lambda - (x_1^2 + x_2^2)], \frac{dx_2}{dt} = x_1 + x_2[\lambda - (x_1^2 + x_2^2)], \quad (26)$$

the linearization around the equilibrium point $(0,0)$ *is*

$$\dot{y} = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} \cdot y. \quad (27)$$

Moreover, the linearized system is: asymptotically stable $\Leftrightarrow \lambda \in (-\infty, 0)$*; stable* $\Leftrightarrow \lambda = 0$*; unstable* $\Leftrightarrow \lambda \in (0, \infty)$.

3.2. Stability Analysis of Rabinovich System

We shall follow the same steps as in the previous case. The algebraic system

$$x_2 x_3 = 0, -x_1 x_3 = 0, x_1 x_2 = 0 \quad (28)$$

gives us the equilibrium point $\left(x_1^*(t), x_2^*(t), x_3^*(t)\right) = (0,0,0)$ of the Rabinovich system. The linearization around the equilibrium point $(0,0,0)$ is

$$\dot{y} = \begin{pmatrix} 0 & x_3 & x_2 \\ -x_3 & 0 & -x_1 \\ x_2 & x_1 & 0 \end{pmatrix}_{(x_1^*, x_2^*, x_3^*)} \cdot y, \quad (29)$$

that is,

$$\dot{y} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot y. \quad (30)$$

By a direct computation, we get $\det(A - sI_3) = -s^3$ (see A as the Jacobian matrix of the function $(x_2 x_3, -x_1 x_3, x_1 x_2)$, computed at $\left(x_1^*(t), x_2^*(t), x_3^*(t)\right) = (0,0.0)$). Solving the equation $\det(A - sI_3) = 0$, we obtain the multiple solution $s_1 = s_2 = s_3 = 0$. Consequently, the linearized system is not asymptotically stable. By $m_a(0) = m_g(0) = 3$, we conclude the linearized system is stable.

Proposition 3.2.1 *For the non-linear system of differential equations (Rabinovich system)*

$$\frac{dx_1}{dt} = x_2 x_3, \frac{dx_2}{dt} = -x_1 x_3, \frac{dx_3}{dt} = x_1 x_2, \quad (31)$$

the linearization around the equilibrium point $(0,0.0)$ *is*

$$\dot{y} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot y. \tag{32}$$

Moreover, the associated linearized system is stable but not asymptotically stable.

3.3. Stability Analysis of Van Der Pol System

Solving the following non-linear algebraic system

$$x_2 = 0, -\varepsilon(x_1^2 - 1)x_2 - x_1 = 0, \tag{33}$$

we get the equilibrium point $\left(x_1^*(t), x_2^*(t)\right) = (0,0)$ of the van der Pol system. The associated linearized system (around the equilibrium point) is

$$\dot{y} = \begin{pmatrix} 0 & 1 \\ -2\varepsilon x_1 x_2 - 1 & -\varepsilon(x_1^2 - 1) \end{pmatrix}_{(x_1^*, x_2^*)} \cdot y, \tag{34}$$

that is,

$$\dot{y} = \begin{pmatrix} 0 & 1 \\ -1 & \varepsilon \end{pmatrix} \cdot y. \tag{35}$$

By direct computation, we get $\det(A - sI_2) = s^2 - \varepsilon s + 1 = s^2 - \mathrm{Tr}(A)s + \det(A)$ (see A as the Jacobian matrix of the function $(x_2, -\varepsilon(x_1^2 - 1)x_2 - x_1)$, computed at $\left(x_1^*(t), x_2^*(t)\right) = (0,0)$). We get $\Delta \doteq \det(A - sI_2) = \varepsilon^2 - 4$. There are three cases:

a) $\Delta = 0 \Leftrightarrow \varepsilon = \pm 2 \Leftrightarrow s_1 = s_2 = \frac{\varepsilon}{2} = \pm 1$. So, for $\varepsilon = -2$ the associated linearized system is asymptotically stable, and for $\varepsilon = 2$ it is unstable.

b) $\Delta > 0 \Leftrightarrow \varepsilon \in (-\infty, -2) \cup (2, +\infty)$. The solutions are given by $s_{1,2} = \frac{\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}$. For $\varepsilon > 2$ the system is unstable (the both solutions do not have negative real part). For $\varepsilon < -2$ the system is asymptotically stable (the both solutions have negative real part).

c) $\Delta < 0 \Leftrightarrow \varepsilon \in (-2,2)$. The solutions are given by $s_{1,2} = \frac{\varepsilon \pm i\sqrt{4 - \varepsilon^2}}{2}$. For $\varepsilon \in (-2,0)$ the linearized system is asymptotically stable. For $\varepsilon \in (0,2)$ the linearized system is unstable. Finally, for $\varepsilon = 0$ ($s_{1,2} = \pm i$) we get $m_a(\pm i) = m_g(\pm i) = 1$, that is the associated linearized system is stable.

Proposition 3.3.1 *For the non-linear system of differential equations (van der Pol system)*

$$\frac{dx_1}{dt} = x_2, \frac{dx_2}{dt} = -\varepsilon(x_1^2 - 1)x_2 - x_1, \tag{36}$$

the linearization around the equilibrium point $(0,0)$ *is*

$$\dot{y} = \begin{pmatrix} 0 & 1 \\ -1 & \varepsilon \end{pmatrix} \cdot y. \tag{37}$$

Moreover, the linearized system is: asymptotically stable $\Leftrightarrow \varepsilon \in (-\infty, 0)$*; stable* $\Leftrightarrow \varepsilon = 0$*; unstable* $\Leftrightarrow \varepsilon \in (0, \infty)$.

3.4. Stability Analysis of the Phytoplankton Growth Model

Solving the following non-linear algebraic system

$$1 - x_1 - \frac{x_1 x_2}{4} = 0, (2x_3 - 1)x_2 = 0, \frac{x_1}{4} - 2x_3^2 = 0, \tag{38}$$

we get the equilibrium points $\left(x_1^*(t), x_2^*(t), x_3^*(t)\right) = \left(2, -2, \frac{1}{2}\right)$, or $\left(1,0,-\frac{\sqrt{2}}{4}\right)$, or $\left(1,0,\frac{\sqrt{2}}{4}\right)$ of the Phytoplankton Growth Model. The associated linearized systems (around the equilibrium points) are

$$\dot{y} = \begin{pmatrix} -1 - \frac{x_2}{4} & -\frac{x_1}{4} & 0 \\ 0 & 2x_3 - 1 & 2x_2 \\ \frac{1}{4} & 0 & -4x_3 \end{pmatrix}_{(x_1^*, x_2^*, x_3^*)} \cdot y, \tag{39}$$

that is,

$$\dot{y} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -4 \\ \frac{1}{4} & 0 & -2 \end{pmatrix} \cdot y, \dot{y} = \begin{pmatrix} -1 & -\frac{1}{4} & 0 \\ 0 & -\frac{\sqrt{2}}{2} - 1 & 0 \\ \frac{1}{4} & 0 & \sqrt{2} \end{pmatrix} \cdot$$

$$y\dot{y} = \begin{pmatrix} -1 & -\frac{1}{4} & 0 \\ 0 & \frac{\sqrt{2}}{2} - 1 & 0 \\ \frac{1}{4} & 0 & -\sqrt{2} \end{pmatrix} \cdot y. \tag{40}$$

By a direct computation, we get $\det(A - sI_3) = 2s^3 + 5s^2 + 2s - 1$ (see A as the Jacobian matrix of the function $\left(1 - x_1 - \frac{x_1 x_2}{4}, (2x_3 - 1)x_2, \frac{x_1}{4} - 2x_3^2\right)$, computed at $\left(x_1^*(t), x_2^*(t), x_3^*(t)\right) = \left(2, -2, \frac{1}{2}\right)$). Solving the equation $\det(A - sI_3) = 0$, we obtain the solutions $s_1 = -1, s_2 = \frac{-3+\sqrt{17}}{2}, s_3 = \frac{-3-\sqrt{17}}{2}$, so the associated linearized system is unstable. For the equilibrium point $\left(1,0,-\frac{\sqrt{2}}{4}\right)$, we get $\det(B - sI_3) = -2s^3 + \left(4 - \sqrt{2}\right)s^2 + 3\sqrt{2}s + 2 + 2\sqrt{2}$ (see B as the Jacobian matrix of the function $\left(1 - x_1 - \frac{x_1 x_2}{4}, (2x_3 - 1)x_2, \frac{x_1}{4} - 2x_3^2\right)$, computed at $\left(x_1^*(t), x_2^*(t), x_3^*(t)\right) = \left(1,0,-\frac{\sqrt{2}}{4}\right)$). The equation $\det(B - sI_3) = 0$ gives the solutions $s_1 = -1, s_2 = -1 - \frac{\sqrt{2}}{2}, s_3 = \sqrt{2}$. Consequently, the associated linearized system is unstable. For the equilibrium point $\left(1,0,\frac{\sqrt{2}}{4}\right)$, we get $\det(C - sI_3) = -2s^3 + \left(-4 - \sqrt{2}\right)s^2 - 3\sqrt{2}s + 2 - 2\sqrt{2}$ (see C as the Jacobian matrix of the function $\left(1 - x_1 - \frac{x_1 x_2}{4}, (2x_3 - 1)x_2, \frac{x_1}{4} - 2x_3^2\right)$, computed at $\left(x_1^*(t), x_2^*(t), x_3^*(t)\right) = \left(1,0,\frac{\sqrt{2}}{4}\right)$). The equation $\det(C - sI_3) = 0$ gives the solutions $s_1 = -1, s_2 = -1 + \frac{\sqrt{2}}{2}, s_3 = -\sqrt{2}$. Consequently, the associated linearized system is asymptotically stable.

Proposition 3.4.1 *For the non-linear system of differential equations (Phytoplankton Growth Model)*

$$\frac{dx_1}{dt} = 1 - x_1 - \frac{x_1 x_2}{4}, \frac{dx_2}{dt} = (2x_3 - 1)x_2, \frac{dx_3}{dt} = \frac{x_1}{4} - 2x_3^2, \tag{41}$$

the linearizations around the equilibrium points

$$\left(2,-2,\frac{1}{2}\right),\left(1,0,-\frac{\sqrt{2}}{4}\right),\left(1,0,\frac{\sqrt{2}}{4}\right) \qquad (42)$$

are

$$\dot{y}=\begin{pmatrix}-\frac{1}{2} & -\frac{1}{2} & 0\\ 0 & 0 & -4\\ \frac{1}{4} & 0 & -2\end{pmatrix}y,\ \dot{y}=\begin{pmatrix}-1 & -\frac{1}{4} & 0\\ 0 & -\frac{\sqrt{2}}{2}-1 & 0\\ \frac{1}{4} & 0 & \sqrt{2}\end{pmatrix}y,\ \dot{y}=\begin{pmatrix}-1 & -\frac{1}{4} & 0\\ 0 & \frac{\sqrt{2}}{2}-1 & 0\\ \frac{1}{4} & 0 & -\sqrt{2}\end{pmatrix}y. \qquad (43)$$

Moreover, the associated linearized systems are: unstable (in the case of the equilibrium points $\left(2,-2,\frac{1}{2}\right),\left(1,0,\frac{\sqrt{2}}{4}\right)$) and asymptotically stable (in the case of the equilibrium point $(1,0,\frac{\sqrt{2}}{4})$.

4. Conclusions

In this paper, we considered some special differential systems (taken from the literature) and we studied the associated geometric dynamics. Taking into account the winds theory, we succeeded to derive some second-order differential equations which permit us to describe the winds produced by the special flows (Hopf, Rabinovich, etc.) and Euclidean metric. Also, a stability analysis of considered models is provided.

Acknowledgements

We would like to thank Professor Constantin Udriste for bringing into our attention this subject and for valuable discussions during the preparation of our work.

References

[1] R. Aris, Vectors, Tensors and the Basic Equations of Fluid Flow, Dover Publ. Inc., New York, 1989.

[2] V. I. Arnold, The Hamiltonian nature of the Euler equations of rigid body dynamics and of ideal fluid, Uspehi Matem. Nauk., XXIV, 3, 147, pp. 225-246, 1969.

[3] A. J. Chorin, J. E. Marsden, A Mathematical Introduction to Fluid Mechanics, Springer-Verlag, New York, 2000.

[4] S. Comsa, G. Cosovici, M. Craciun, J. Cringanu, N. Dumitriu, P. Matei, I. Rosca, O. Stanasila, A. Toma, C. Udriste (coordinator), Variational Calculus (in romanian), StudIS Publishing House, Iasi, 2013.

[5] D. Isvoranu, C. Udriste, Fluid flow versus Geometric Dynamics, BSG Proceedings, 13, pp. 70-82, 2006.

[6] G. Schubert, C. C. Counselman, J. Hansen, S. S. Limaye, G. Pettengill, A. Seiff, I. I. Shapiro, V. E. Suomi, F. Taylor, L. Travis, R. Woo, R.E. Young, Dynamics, winds, circulation and turbulence in the atmosphere of venus, Space Science Reviews, 20, 4, pp. 357-387, 1977.

[7] S. Treanta, On multi-time Hamilton-Jacobi theory via second order Lagrangians, U. P. B. Sci. Bull., Series A: Appl. Math. Phys., 76, 3, pp. 129-140, 2014.

[8] C. Udriste, Geometric Dynamics, Mathematics and Its Applications, 513, Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.

[9] C. Udriste, Geodesic motion in a gyroscopic field of forces, Tensor, N. S., 66, 3, pp. 215-228, 2005.

[10] C. Udriste, A. Udriste, From flows and metrics to dynamics and winds, Bull. Cal. Math. Soc., 98, 5, pp. 389-394, 2006.