

# Quantum Field Theory on Noncommutative Curved Space-times and Noncommutative Gravity

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**Abstract:** In the noncommutative gauge-theoretical formulation of Langman and Szabo, apparently it appears that the torsion generated there is a generalized one i.e. it may contain vector, axial vector and tensor components. However, when we transcribe the noncommutative gauge theory in terms of the Maxwell gauge theory using the Seiberg-Witten correspondence, we have noted that upto the first order in the noncommutative parameter, this effectively can be taken to induce a change in chiral anomaly and hence the associated torsion should be an axial vector one. The noncommutative gauge symmetries give a very natural and explicit realizations of the mixing of space-time and internal symmetries which is a characteristic feature of the conventional gauge theory of gravity. The gauge fields of the dimensionally reduced noncommutative Yang-Mills theory map onto a Weitzenböck space time and a teleparallel theory of gravity arises as the zero curvature reduction of a Poincare gauge theory which induces an Einstein-Cartan space-time characterized by connections with both nonvanishing torsion and curvature. However, the teleparallelism equivalent of general relativity involves all the components of torsion. The chiral anomaly in the Einstein-Cartan space  $U_4$  is characterized by the topological invariants like Pontryagin density as well as the Nieh-Yan density when the latter term involves the length scale governed by the measure of noncommutativity of space points. It is shown that we have discussed the equivalence of this formalism with noncommutative  $U(1)$  Yang Mills theory.

**Keywords:** Noncommutative Space-Time, Teleparallel Gravity, Berry Phase

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## 1. Introduction

A local field theory can be obtained by restricting noncommutative Yang-Mills fields such that the symmetry group contains diffeomorphism invariance. Noncommutative gauge symmetries give a very natural and explicit realizations of the mixing of space-time and internal symmetries which is a characteristic feature of the conventional gauge theory of gravity. The gauge fields of the dimensionally reduced noncommutative Yang-Mills theory map onto a Weitzenböck spacetime and a teleparallel theory of gravity arises as the zero curvature reduction of a Poincare gauge theory which induces an Einstein-Cartan space-time characterized by connections with both nonvanishing torsion and curvature. It has been sharp out that in the manifold  $M_4 \times Z_2$  where the discrete space corresponds to a direction-vector,  $Z_2$  symmetry breaking leads to chiral anomaly which is responsible for the topological origin of mass and describes electroweak theory in a consistent way. Thus as in the case of

a noncommutative manifold where  $Z_2$  is a two-point space there appears to be a connection between gravity and electroweak theory in this formalism this is achieved through the realization of chiral anomaly and torsion. Here we shall study the noncommutative space-time having the structure  $M_4 \times Z_2$  where  $Z_2$  is not a two point space but describes a 'direction-vector' attached to a space-time point. It is shown that this also leads to a torsioned space-time such that general relativity is described as a teleparallel gravity. However unlike noncommutative Yang-Mills theory which involves the symplectic area preserving diffeomorphism, here we have deformation of the symplectic structure. Indeed this deformation of the symplectic structure is related to the Berry phase which is associated with chiral anomaly and this leads to a torsioned space-time.

In section 2 we shall discuss teleparallel gravity in the framework of this geometry and equivalence with the formulation of noncommutative gauge theory.

## 2. Noncommutative Gauge Theory, Chiral Anomaly and Torsion

Langmann and Szabo [1] have shown how noncommutative  $U(1)$  Yang-Mills theory on the space  $R^n \times R^n$  can generate a theory of gravitation on  $R^n$ . Let  $G^{AB}$  be a flat metric on  $R^{2n}$ . The standard action for Yang-Mills theory is defined by

$$S = \frac{1}{2} \int_{R^{2n}} d^{2n} \xi \sqrt{\det G} G^{AA'} G^{BB'} \hat{F}_{AB} * \hat{F}_{A'B'}(\xi) \quad (1)$$

where  $\xi$  denotes local coordinates of  $R^{2n}$  given by  $\xi = (x^\mu, y^a)$  where  $\mu, a = 1, 2, \dots, n$ . Thenoncommutative field strength is given by

$$\hat{F}_{AB} = \partial_A \hat{A}_B - \partial_B \hat{A}_A + e[\hat{A}_A * \hat{A}_B - \hat{A}_B * \hat{A}_A] \quad (2)$$

For completeness, we recapitulate here some of the crucial features of their observation. Thenoncommutativity parameters are taken to be of the block form

$$S = \frac{1}{2} \int_{R^{2n}} d^{2n} \xi W^{AA' BB'}(\xi) \hat{F}_{AB} * \hat{F}_{A'B'}(\xi) \quad (3)$$

For completeness, we recapitulate here some of the crucial features of their observation. Thenoncommutativity parameters are taken to be of the block form

$$\theta^{AB} = \begin{bmatrix} \theta^{\mu\nu} & \theta^{\mu b} \\ \theta^{va} & \theta^{ab} \end{bmatrix} \text{ with } \theta^{\mu\nu} = \theta^{ab} = 0 \quad (4)$$

The flat metric of  $R^{2n}$  is taken to be

$$g^{AB} = \begin{bmatrix} \eta^{\mu\nu} & \eta^{\mu b} \\ \eta^{va} & \eta^{ab} \end{bmatrix} \text{ with } \theta^{\mu\nu} = \theta^{ab} = 0 \quad (5)$$

The linear subspace of smooth function  $\alpha$  on  $R^{2n}$  are taken to be linear in the coordinates

$$\alpha(\xi) = \alpha_a(x) y^a \quad (6)$$

We now define

$$g = Vect(R^n)$$

$$X_\alpha = -\theta^{\mu a} \alpha_a \frac{\partial}{\partial x^\mu} \quad (7)$$

so that we can have a representation of the Lie algebra

$$[X_\alpha, X_\beta] = X_{[\alpha, \beta]}, \forall \alpha, \beta \in g \quad (8)$$

Here,  $g$  can be identified with the Lie algebra of connected diffeomorphism of  $R^n$ . Defining

$$f^1(\xi) = f_a(x) y^a \text{ and } g^0(\xi) = g(x) \quad (9)$$

we have the star product rule

$$f^1 * g^0(\xi) = g(x) f_a(x) y^a - \frac{1}{2} \theta^{\mu a} f_a(x) \partial_\mu g(x) \quad (10)$$

From this we can consider the gauge transformation rule

$$\partial \phi = -\theta^{\mu a} \alpha_a \partial_\mu \phi \quad (11)$$

Under a global transformation  $x^\mu \rightarrow x^\mu + e^\mu$ , the scalar fields transform infinitesimally as

$$\phi(x) \rightarrow \phi(x) + e^\mu \partial_\mu \phi(x) \quad (12)$$

Since

$$\partial_\mu \phi(x) = -(\theta^{-1})_{a\mu} [y^a, \phi] * (x) \quad (13)$$

we can identify  $y^a$  with the holonomic derivative generators  $-\theta^{\mu a} \partial_\mu$  of the  $n$ -dimensional translational group  $T$  of  $R^n_x$ . This helps us to write the covariant derivative

$$\nabla_\mu = \partial_\mu - e \omega_{\mu a} \theta^{va} \partial_v \quad (14)$$

where  $\omega_{\mu a}$  are gauge fields corresponding to the gauging of the translation group. Now defining curvature of the gauge field

$$\hat{F}_{\mu\nu}(\xi) = \Omega_{\mu\nu a} y^a \quad (15)$$

where

$$\Omega_{\mu\nu a} = \partial_\mu \omega_{\nu a} - \partial_\nu \omega_{\mu a} + e \theta^{\lambda b} (\omega_{\nu b} \partial_\lambda \omega_{\mu a} - \omega_{\mu b} \partial_\lambda \omega_{\nu a}) \quad (16)$$

and replacing  $y^a$  by  $-\theta^{va} \partial_v$  we note that the covariant derivative  $\nabla_\mu$  defines a nonholonomic basis of the tangent bundle with nonholonomicity given by the field strength tensor. The commutator of covariant derivatives is given by

$$(\nabla_\mu, \nabla_\nu) = T_{\mu\nu}^\lambda \nabla_\lambda \quad (17)$$

This commutation relation identifies  $T_{\mu\nu}^\lambda$  or equivalently noncommutative gauge field strength  $\Omega_{\mu\nu a}$  as the torsion tensor fields of vacuum space-time induced by the presence of a gravitational field. This induces a teleparallel structure of space-time through the Weitzenböck connection with nonvanishing torsion and vanishing curvature. Indeed the torsion  $T_{\mu\nu}^\lambda$  measures the noncommutativity of displacement of points in the flat space-time  $R^n_x$ . It is dual to the Riemann curvature tensor which measures noncommutativity of vector displacements in a curved space-times. Now we consider the equivalence of this formalism of gravity from noncommutative  $U(1)$  Yang-Mills field with that attained

from the noncommutative geometry  $M_4 \times Z_2$  where spacetime coordinates are extended by  $SL(2, C)$  gauge fields and torsion is obtained from chiral anomaly. Indeed we can identify the position and momentum operators as

$$\begin{aligned}\frac{Q_\mu}{\omega} &= -i\left(\frac{\partial}{\partial p_\mu} + A_\mu\right) \\ \frac{P_\mu}{\omega} &= -i\left(\frac{\partial}{\partial q_\mu} + A_\mu\right)\end{aligned}\quad (18)$$

where  $\omega$  is the dimensionless variable,  $\omega = \frac{\hbar}{lmc}$ . Here  $q_\mu$  is the space-time point in Minkowski space  $M_4$  and  $p_\mu$  its conjugate representing the momentum variable. We have pointed out that the gauge theoretical extension of space-time and momentum coordinate given by eqn.(18) effectively deforms the symplectic structure. Indeed from equation (18) we note that the symplectic form of the phase space will now be given by

$$\Omega = \frac{1}{2} g^{ij} dp_i \wedge dq_j \quad (19)$$

the symplectic form is given by eqn.(19) where the matrix  $\Delta^{ij}$  is associated with the gauge field strength. Indeed by replacing the indices  $i, j$  by  $\mu, \nu$ , we may associate it with the field strength  $F_{\mu\nu}(\vec{\xi})$  with  $\vec{\xi} = (\vec{x}, \vec{P})$ . Now following eqn.(15) we write

$$F_{\mu\nu}(\vec{\xi}) = \Omega_{\mu\nu a} \theta^{\lambda a} \partial_\lambda \quad (20)$$

where  $\Omega_{\mu\nu a}$  is the field strength associated with the gauge field  $\omega_{\mu a}$  given by eqn.(16). It is noted that  $\theta^{\nu a}$  here effectively corresponds to the matrix  $j^{ij}$

$$g^{ij} = j^{ij} + \hbar \Delta^{ij} \quad (21)$$

in eqn.(21) and represents the usual symplectic structure. To trace the nature of the gauge field  $\omega_{\mu a}$ , we decompose it in the form

$$\omega_{\mu a}(x) = W_\mu \bar{\theta}^{av} x_\nu \quad (22)$$

Where

$$\bar{\theta}^{av} = i\mu \epsilon^{av\lambda} \frac{x_\lambda}{r^3} \quad (23)$$

$\epsilon^{av\lambda}$  being the fundamental Levi-Civita tensor. In eqn. (23)  $\mu$  appears as the monopole strength. It may be noted that  $\bar{\theta}^{av}$  here is essentially related to the noncommutativity of momentum space. Indeed when we consider that the symplectic matrix is not a constant one but dependent

on  $x$ , it violates the associativity property. The failure of the Jacobi identity

$$[x^i, [x^j, x^k]] + [x^j, [x^k, x^i]] + [x^k, [x^i, x^j]] \neq 0 \quad (24)$$

implies the presence of the dual monopole which is analogous to the failure of the Jacobi identity between momentum implying the presence of the Dirac monopole [2]. Thus replacing the constant symplectic matrix by the gauge field strength  $F_{\mu\nu}$  effectively implies that the corresponding gauge field is operative in the presence of a magnetic monopole. This essentially indicates that the gauge field  $W_\mu$  in eqn. (22) should be a non-Abelian one. This follows from the fact that we can associate the monopole strength  $\mu$  with the Pontryagin index  $q$  associated with the  $\theta$ -term in non-Abelian gauge field. In fact, we have the relation [3]

$$q = 2\mu = \frac{1}{16\pi^2} \text{Tr} \int *F_{\mu\nu} F_{\mu\nu} d^4x \quad (25)$$

The hidden Abelian gauge field in a non-Abelian gauge theory associated with the topological  $\theta$ -term  $*F_{\mu\nu} F_{\mu\nu}$  in the Lagrangian may be viewed as if in the gauge orbit space, the position of a particle is indicated by  $A$  (non-Abelian gauge potential) moving in the space  $U$  of non-Abelian gauge potentials under the influence of an Abelian electromagnetic potential [4, 5]. We can write

$$A = g^{-1} dg + g^{-1} a g \quad (26)$$

where  $A = A_\mu dx^\mu$ ,  $a = a_\mu dx^\mu$  and  $g(x)$  is a matrix associated with the gauge transformation. The space of gauge orbits  $U/G$  where  $G$  denotes the space of local transformations of  $g(x)$  consists of the point  $a(x)$ . Recalling that  $\Pi_3(G) = Z$  for all simple non-Abelian groups  $G$  and  $\Pi_2(G) = 0, \Pi_n(U) = 0$  for all  $n$ , we have

$$\Pi_n(U/G) = \Pi_{n-1}(G), n \geq 1 \quad (27)$$

That means in 3 + 1 dimension

$$\Pi_1(U/G) = \Pi_0(G) = \Pi_3(G) = Z \quad (28)$$

This equality  $\Pi_0(G) = \Pi_3(G) = Z$  follows from the condition that the gauge transformation  $g(x)$  approaches constant independent of the direction of  $x$  as  $x \rightarrow \infty$ . Thus  $U/G$  is multiply connected and has the topology of a ring and the corresponding field strength corresponds to a vortex line which is topologically equivalent to a magnetic flux line. Now we note that when a particle moves in the presence of a magnetic monopole, the angular momentum is given by

$$\vec{J} = \vec{r} \times \vec{p} - \mu \hat{r}, \mu = \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \dots \quad (29)$$

In case  $|\mu| = \frac{1}{2}$ , we can associate this with a spinor and in

view of this we consider  $W_\mu$  as an  $SU(2)$  gauge field. In 3+1 dimension, the group structure can be generalized to  $SL(2, C)$  SL. Thus we find that the torsion generated by the noncommutative gauge field theory may be considered to be equivalent to the torsion generated by the  $SL(2, C)$  gauge theory in noncommutative space  $M_4 \times Z_2$ .

We have observed in the previous section that the noncommutative manifold  $M_4 \times Z_2$  induces torsion through chiral anomaly which is related to the Berry Phase associated with the deformation of the symplectic structure. The formulation of noncommutative  $U(1)$  Yang-Mills theory also implicitly induces the change in chiral anomaly. However, in this case we have area preserving diffeomorphism. Indeed, the association of the gauge field  $\omega_{\mu a}$  with the gauge field  $W_\mu$  in presence of a monopole related to the  $\theta$ -term in non-Abelian gauge field theory effectively relates the torsion term associated with the noncommutative  $U(1)$  gauge field theory with the chiral anomaly. In fact, the topological term  $Tr * F_{\mu\nu} F_{\mu\nu}$  effectively corresponds to the divergence of the axial vector current and hence we may view that the torsion is of axial vector in nature. This can be shown in a more explicit form from the following considerations. Thenoncommutative field strength constructed from the potential  $\hat{A}_\mu$  is given by

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + g[\hat{A}_\mu * \hat{A}_\nu - \hat{A}_\nu * \hat{A}_\mu] \quad (30)$$

when we have the Lagrangian

$$\hat{L} = -\frac{1}{4} \hat{F}_{\mu\nu} * \hat{F}^{\mu\nu} \quad (31)$$

In terms of the conventional Maxwell term

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (32)$$

it turns out that to first order in  $\theta^{\alpha\beta}$  we can express  $\hat{A}_\mu$  and  $\hat{F}_{\mu\nu}$  as follows [6]

$$\hat{A}_\mu = A_\mu - \frac{1}{2} \theta^{\alpha\beta} A_\alpha (\partial_\beta A_\mu + F_{\beta\mu}) \quad (33)$$

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \theta^{\alpha\beta} A_\alpha \partial_\beta F_{\mu\nu} \quad (34)$$

where  $g$  is absorbed in  $\theta$ . The tensor  $\theta^{\alpha\beta}$  is associated with the star product which is defined as

$$(f * g)(x) = e^{\frac{i}{2} \theta^{\alpha\beta} \partial_\alpha \partial_\beta} f(x) g(x) \Big|_{x=x'} \quad (35)$$

Apart from the total derivative term, the Lagrangian  $\hat{L}$  is given by

$$\hat{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{8} \theta^{\alpha\beta} F_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \theta^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} F^{\mu\nu} + O(\theta^2) \quad (36)$$

It is well known that star product effectively involves a background magnetic field [7] and so the second and the third terms in equations (34) and (36) correspond to the interaction of this background field with the Maxwell field. When we consider the interaction of the chiral current with the noncommutative gauge field having the field strength to the first order in  $\theta$ , we have

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \theta^{\alpha\beta} A_\alpha \partial_\beta F_{\mu\nu} = F_{\mu\nu} + \tilde{F}_{\mu\nu} \quad (37)$$

We note that the chiral anomaly will be modified as

$$\frac{1}{8\pi^2} Tr \int [ * F_{\mu\nu} F_{\mu\nu} + * F_{\mu\nu} \tilde{F}_{\mu\nu} + * \tilde{F}_{\mu\nu} \tilde{F}_{\mu\nu} ] \quad (38)$$

This implies that the induced background magnetic field associated with the star product effectively changes the chiral anomaly. In the noncommutative gauge-theoretical formulation of Langman and Szabo, apparently it appears that the torsion generated there is a generalized one i.e. it may contain vector, axial vector and tensor components. However, when we transcribe the noncommutative gauge theory in terms of the Maxwell gauge theory using the Seiberg-Witten correspondence, we have noted that upto the first order in the noncommutative parameter  $\theta$ , this effectively can be taken to induce a change in chiral anomaly and hence the associated torsion should be an axial vector one. However, the teleparallelism equivalent of general relativity involves all the components of torsion. In fact, if we denote the Lagrangian in teleparallel gravity as [8]

$$L_T = -\frac{1}{2l^2} T^\alpha \wedge * (-^{(1)}T_\alpha + 2^{(2)}T_\alpha + \frac{1}{2}^{(3)}T_\alpha) \quad (39)$$

where  $^{(1)}T_\alpha$ ,  $^{(2)}T_\alpha$ ,  $^{(3)}T_\alpha$  corresponds to tensor, vector (trace) and axial vector components of torsion, its equivalence with general relativity arises from the geometrical identity

$$-\frac{1}{2} R^{\alpha\beta} \wedge \eta_{\alpha\beta} + \frac{1}{2} R^{\{\alpha\beta} \wedge \eta_{\alpha\beta} + l^2 L_T = d(e^\alpha \wedge * T_\alpha) \quad (40)$$

where  $R^{\alpha\beta}$  is the curvature associated with the Einstein-Cartan space-time,  $R^{\{\alpha\beta}$  is the curvature in Riemannian space-time and  $\eta_{\alpha\beta} = *(e_\alpha \wedge e_\beta)$ . From this we note that in Weitzenböck space-time with vanishing Riemann-Cartan space curvature  $R^{\alpha\beta} = 0$  the Lagrangian  $L_T$  is upto a boundary term equivalent to the Einstein-Hilbert Lagrangian. As the teleparallel Lagrangian involves all the components of torsion with specific coefficients, only the axial vector component of torsion can contribute if by any constraint the contributions of the vector and tensor components in the action cancel.

### 3. Discussion

In our present framework, we observe that  $Z_2$  symmetry breaking leads to chiral anomaly. In recent paper [9-11] it has been shown that chiral anomaly gives rise to the topological origin of mass. Besides chiral anomaly is associated with the nonvanishing value of the Berry phase factor [12, 13]. This is linked up with the conserved charge through Dirac quantization condition. So we observed that the symmetry breaking in the noncommutative space-time given by  $M_4 \times Z_2$  when the discrete space is considered as the internal space not only generates mass but also a locally defined conserved charge. Recently it has been pointed out that the N-Y density may be taken to arise from the noncommutativity of space when the space-time manifold is  $M_4 \times Z_2$ , we note that gravitational constant may be considered to be a manifestation of the noncommutative geometry. It may be remarked that in the noncommutative  $U(1)$  Yang-Mills field theoretical formulation also the gravitational constant is found to be associated with the gauge coupling constant and noncommutativity scale [1].

### 4. Conclusion

Finally We may add that in flat space as the manifold  $M_4 \times Z_2$  is found to be associated with the quantization of a fermion when the  $Z_2$  symmetry breaking leads to chiral anomaly which is responsible for the origin of mass [14], it is very natural to link up gravitation with the noncommutativity of space and in this way the association of gravitation with quantum mechanics becomes relevant [15, 16]. Thus the noncommutative geometry paves the path to have a reconciliation of general relativity and quantum mechanics.

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